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Ran Lanzet

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Whittle vs. Cantor on the Size of Infinite Sets

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I examine several arguments by Bruno Whittle against the Cantorian conception of the size of infinite sets. I find that none of them succeed.

Sets A and B are the same size—A has exactly as many elements as B does—iff there is a bijection between them; A is at least as large as B—has at least as many elements as B does—iff there is an injection from B to A. This conception of set size has become standard following Cantor. What Cantor's theorem means, on this conception, is that the powerset of every set *A* is larger than *A* (see, e.g., Enderton 1977, 132; Hrbacek and Jech 1999, 65; Smullyan and Fitting 1996, 7–8). Thus interpreted, the theorem entails that there are different sizes of infinity (assuming, of course, that there is at least one infinite set and that that set has a powerset).

Whittle (2015a, 2015b, 2018) objects to this standard conception. He holds that we do not in fact have good reason for believing either of the following two principles, and that we are thus not in a position to know either of them or to know that there are infinite sets of different sizes:

> Size→Function
> For any sets *A* and *B*, if *A* is the same size as *B*, then there is a bijection from *A* to *B*.
>
> Size*→Function
> There is some "size-like" property—a property *similar* to size—size* such that, for any sets *A* and *B*, if *A* is the same size* as *B*, then there is a bijection from *A* to *B*.

Whittle puts forward a series of arguments that aim to establish this. Five of his arguments purport to refute the following widely accepted theses:

THESIS 1. There being a bijection between A and B is what it is for A and B to be the same size.[1,2]

THESIS 2. The notion of cardinality defined in set theory is at least "size-*like*": it is at least similar to our ordinary notion of size (of sets) and perhaps a natural generalization of the notion of finite size.

THESIS 3. Cantor's theorem establishes that there are infinite sets of different sizes.[3]

Here, I examine Whittle's arguments against theses 1 and 2; I argue that none of them succeed as refutations of either of those theses. The arguments, appearing in their fully developed form in Whittle (2018), are:

(i) an argument against THESIS 1 based on its interpretation as stating that "c is the same size as d" and "There is a bijection from c to d" express the same structured proposition;
(ii) an argument against the same thesis, based on what would be true in a mathematically-impossible situation in which there are no functions from certain sets;
(iii) a Benacerraf-style challenge for accounts of size in terms of functions—accounts that may be offered in support of THESIS 1;
(iv) an objection to THESIS 2 based on (ii).

Whittle also argues directly against THESIS 3 in his (2015a) and (2015b). I examine and reject his argument against this thesis in (manuscript).

1 Whittle targets this thesis and THESIS 2 below since they can be offered as grounds for SIZE→FUNCTION and SIZE*→FUNCTION.

2 Some may consider THESIS 1 too strong and prefer a more tolerant approach, on which the identification of size with cardinality-as-defined-in-set-theory is just one of several legitimate options. However, if this identification is indeed untenable, as some of Whittle's arguments purport to establish, then it is unclear how one could legitimately adopt it—even tolerantly. I thank an anonymous referee for drawing my attention to this point.

3 Whittle also attacks the following theses, which, I believe, are not as popular as the ones I mentioned in the main text: We can infer that SIZE→FUNCTION holds for sets in general from the fact that it holds for finite sets; we can infer SIZE→FUNCTION by Inference to the Best Explanation, since size differences are the only explanation of the absence of a bijection between two given sets; we are entitled to consider SIZE→FUNCTION as a basic mathematical truth; we can argue for SIZE→FUNCTION inductively, based on the consequences that it allows us to derive. I will not discuss Whittle's arguments against those additional theses here.

If I am right in rejecting the arguments against theses 1–3, then Whittle fails to establish his claim that we are not in a position to know that there are different sizes of infinity. For these three theses, if correct, provide routes to such knowledge (this is, indeed, why Whittle attacks them); and this is so whether or not some additional routes are successfully blocked by those of Whittle's arguments that I will not consider here.[4]

Although my treatment of Whittle's arguments can be considered a defense of the Cantorian view of size, I will not offer positive arguments for this view or argue that other views, incompatible with it, are incorrect or unjustified; my point is only that Whittle fails to establish this for Cantor's view.

Sections 1–4 below are each dedicated to the examination of one of the arguments (i)–(iv). I briefly conclude in section 5.

1 Structured Propositions

Consider the following two sentences:

> SIZE
> *c* is the same size as *d*.
>
> FUNCTION
> There is a bijection from *c* to *d*.

According to Whittle, the most straightforward interpretation of THESIS 1 is this:

(*) SIZE and FUNCTION express the same proposition.[5]

Whittle thinks of propositions here as structured, Russellian propositions. The propositions that SIZE and FUNCTION seem to express are, according to him,

$$(p_S)\ \langle \exists! P[\text{Size}(P) \wedge c \text{ has } P] \ \wedge\ \exists! Q[\text{Size}(Q) \wedge d \text{ has } Q] \wedge \exists R[\text{Size}(R) \wedge c \text{ has } R \wedge d \text{ has } R]\rangle$$

and

4 See footnote 3.

5 An alternative interpretation of THESIS 1 would be that SIZE and FUNCTION describe the same feature of reality. Whittle's argument against THESIS 1 on *this* reading is the one considered in the next section.

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(p_F) $\langle \exists f \;\; f$ is a bijection from c to $d \rangle$,

respectively.

Whittle now suggests that each of the steps in the following argument against (*) is at least very plausible, or prima facie correct (Whittle 2018, 855–856):

(1) (p_S) and (p_F) are about very different sorts of things: (p_S) is about a certain sort of property—a size—that c and d have in common; (p_F) is about a function.
(2) If (*) is true, then either both Size and Function express (p_S), or both of them express (p_F).
(3) Most sentences about functions express propositions that are genuinely about functions.
(4) There is no plausible account on which the following sentence expresses a proposition about properties:

    Constant
    There is a constant function from c to d.

(5) The sentences Function and Constant seem to be very similar.
(6) From (5): Function and Constant should express similar propositions—not ones about completely different sorts of things.
(7) From (1), (4), and (6); and also from (1) and (3): Function cannot express (p_S).
(8) Size has the same general form as the following:

    Color
    e is the same color as g.

    Height
    e is the same height as g.

    Sex
    e is the same sex as g.

(9) From (8): Size and sentences like Color, Height, and Sex should express similar propositions—propositions of the same general form.

(10) The propositions expressed by sentences such as COLOR, HEIGHT, and SEX are not about functions.
(11) From (1), (9), and (10): SIZE cannot express (p_F).
(12) From (7) and (11): It is not the case that both SIZE and FUNCTION express (p_S), and it is not the case that both of them express (p_F).
(13) From (2) and (12): (*) is false.

Whittle's argument here relies on the following view of structured propositions:

> PROP
> For each declarative sentence such as SIZE and FUNCTION, there is one unique structured proposition that the sentence can be taken to express; the structure of that proposition, as well as what it is about, is given by the most straightforward formalization of the sentence in higher-order predicate logic.

Without PROP, we seem to have no reason to accept the inference from (5) to (6) or the one from (8) to (9). As I will now argue, however, it is illegitimate to rely on PROP in this context. This is so, since PROP is incompatible with Whittle's reading of "what it is"—a reading on which

- There being a bijection between A and B is what it is for A and B to be the same size

is correctly interpreted as

- SIZE and FUNCTION express the same structured proposition.

PROP, together with this reading, yields absurd consequences, as illustrated by the following example.

Being a man who has never been married is just what it is to be a bachelor. But PROP, together with the aforementioned reading of "what it is," entails that this is not so. To see this, note first that the most straightforward formalizations in predicate logic of the following differ in structure:

(i) x is a bachelor.
(ii) x is a man who has never been married.

(i) and (ii) are most straightforwardly formalized as, respectively:

(iii) Bx
(iv) $M^1 x \wedge \neg\exists y(M^2 xy)$.

This, by Prop, means that (i) and (ii) express different propositions. Hence, on Whittle's interpretation of "what it is," being a never-married man is *not* what it is to be a bachelor.

One can, of course, hold that, in formalizing (i), we should take into account the analysis of "bachelor" as "never-married man" and correspondingly allow the formalization of both (i) and (ii) as (iv). This is, I think, a reasonable approach, but note that it deviates from Prop. Moreover, if a similar deviation is allowed in the case of Size and Function, then Whittle's argument does not go through, as there is no longer reason to accept, e.g., the inference from (8) to (9).

Whittle's approach entails, then, that being a never-married-man is *not* what it is to be a bachelor. Similarly, it entails that:

- There being an isometry that transforms a figure c in the Euclidean plane into a figure d is *not* what it is for c and d to be congruent.
- There being a formal proof of a first-order sentence ϕ from a first-order theory T is *not* what it is for ϕ to be a theorem of T.

Additional, similar examples are not hard to produce.

I conclude that Whittle's argument from structured propositions fails by relying on two incompatible principles: Prop and a reading of "what it is," on which Thesis 1 is correctly interpreted as (*). If Prop is given up, then, as noted above, Whittle's argument seems not to go through; if the aforementioned reading of "what it is" is given up, then the argument does go through, but its conclusion, (13), no longer means that Thesis 1 is false.

2 Sets in an Impossible Situation

Thesis 1, recall, was this: There being a bijection between A and B is what it is for A and B to be the same size. This thesis can be supported by what Whittle calls a *functional account of size*: an account of size properties of sets in terms of functions. According to Whittle, however, no such account can be correct (Whittle 2018, 860–861).

Whittle specifically considers a functional account on which complex properties are "Russellian propositions with gaps" and sizes are properties of the form

⟨There is a bijection from __ to κ⟩,

where κ is a von Neumann cardinal number (i.e., a von Neumann ordinal equipollent with none of its elements). This, of course, is not the only way to characterize size properties in terms of functions. A more natural functional account—and one that avoids commitment to a specific metaphysical account of properties—is, perhaps, this: A size-property is the property of belonging to a given bijection-type[6] (cf. Whittle 2018, fn.15). Whittle explains, however, that his argument can be adapted so as to refute *any* functional account of size, and, in fact, any account of size in terms of things other than those that constitute the set, such as its elements or those elements' parts or elements.

Whittle's argument involves the sets $A = \{0, 1\}$ and $B = \{2, 3\}$ in a (mathematically impossible) situation S that is "exactly like the actual world, except that there are no functions from either set" (Whittle 2018, 860). The argument is this:

(1) In S, there are no bijections from A or from B.
(2) In S, A and B have the same size.
(3) From (1): In S, neither A nor B has any property of the form ⟨There is a bijection from __ to κ⟩.
(4) From (3): If the functional account under consideration is correct, then, in S, neither A nor B has a size.
(5) From (4): If the functional account is correct, then, in S, A and B do *not* have the same size.
(6) From (2) and (5): The functional account is incorrect.

There is a good reason to suspect this argument, if not to reject it outright: Arguments very similar to it, and ones that are not any less appealing, lead to absurd conclusions. Consider, for example, the following account of what a *circle* is:

> CIRCLE
> To be a circle is to be a set of all the points in the Euclidean plane that are at the same given distance r from a given point o.

I submit that this is a *correct* account of what a circle is. But an argument similar to Whittle's leads to the conclusion that this is not so: Let C be a circle,

6 A bijection type here is a class X for which the following condition holds: For some set a, X is the class of all possible sets x such that there is a bijection between x and a.

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and consider an impossible situation S' that is exactly like the actual world except that there is no point at equal distances from all the elements (points) of C. (Reasoning about S' seems to make as much sense as reasoning about S does.) Then:

(1) In S', no point is at equal distances from all the elements of C.
(2) In S', C is a circle. (This seems as plausible as Whittle's premise that A and B have the same size in S.)
(3) From (1): If CIRCLE is correct, then, in S', C is *not* a circle.
(4) From (2) and (3): CIRCLE is incorrect.

(Note that I am not claiming that this is a very convincing argument—just that it is similar to, and not less appealing than, Whittle's argument.)

This "bad company" indicates that something is wrong with Whittle's argument. But what? There are, I believe, two major problems with the argument. First, it is not at all clear that we can make sufficient sense of mathematically impossible situations like S to determine the truth value of statements like Whittle's premise (2) ("In S, A and B have the same size"). Whittle claims that we *are* capable of judging what would be true in S, but it is far from clear that he is correct about that. Whittle also claims that an argument similar to his can be given not in terms of what is true under an impossible hypothesis but, instead, in terms of what is an immediate consequence of the hypothesis. If this is indeed possible, then the resulting (modified) argument would not be any more problematic than a standard reductio. It is unclear, however, how the imagined modification of the argument is supposed to proceed, and Whittle gives no indication of that. At least on the face of it, his argument is very much *un*like a reductio: It is more similar, it seems, to a demonstration of non-entailment using a counterexample; for, rather than deriving an absurdity from an impossible hypothesis, he seems to be relying on judgments made under such a hypothesis in order to reject a universal statement (to the effect that certain properties always coincide).

Second, assuming (for the sake of argument) that we can make sufficient sense of mathematically impossible situations like S, it is unclear why we should think that A and B have the same size in S. Perhaps Whittle holds that this is so since (allegedly) A and B are still of size 2 in S. But it is unclear why we should think that this is so. Especially if, in S, *there is no way of counting the elements of A or those of B*, and there is indeed no way of counting those elements in S, at least if the following jointly hold: (a) a way of counting the elements of, e.g., A is a way of correlating them 1-1 with the elements of {"one", "two"}; (b) a way of thus correlating the elements of these two sets is a *bijection* between them; (c) in S, there are no bijections between A and any other set.

Perhaps Whittle holds that, e.g.,

> SIZE2
> The size of A is 2 in *every* situation, mathematically possible or not, in which A exists.[7]

But this, it seems, cannot serve as a ground for Whittle's premise (2), since an argument similar to his leads to the conclusion that SIZE2 is *false*: Consider a situation S'' that is exactly like the actual world, except that there are no numbers other than 0 and 1. (Reasoning about S'' seems to make as much sense as reasoning about S does.) Then:

(1) It is not the case that, in S'', something equals 2.
(2) If SIZE2 is true, then, in S'', the size of A equals 2.
(3) From (2): If SIZE2 is true, then, in S'', something equals 2.
(4) From (1) and (3): SIZE2 is false.

I am not claiming that (1)–(4) is a convincing argument or that reasoning in this way about S'' makes any sense; I only claim that this is so *by Whittle's standards*. If this claim is correct, then Whittle cannot rely on SIZE2; consequently, premise (2) of his argument remains unfounded.

I conclude that Whittle's argument from S fails as a refutation of functional accounts of size and that it is therefore ineffective against THESIS 1.

3 Benacerraf's Problem

Whittle (2018, 862–863) argues that the specific functional account that featured in his argument from an impossible situation faces a version of "Benacerraf's problem."[8] That functional account, recall, identified sizes with properties of the form

7 He may think that this is so, since the following (presumably) holds in any situation in which A exists: $\exists u \exists v(u \neq v \wedge u \in A \wedge v \in A \wedge \forall w[w \in A \rightarrow (w = u \vee w = v)])$.

8 The original problem, presented in Benacerraf (1965), arises for theories that take the natural numbers to be a particular collection of sets.

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⟨There is a bijection from __ to κ⟩,

where κ is a von Neumann cardinal number. The problem is this: By replacing the cardinal numbers with other, equipollent sets, we can obtain different, incompatible functional accounts of size. Since, moreover, there is no reason to favor any one of those competing accounts over the others, each of them seems arbitrary and therefore, according to Whittle, incorrect.

This problem does not seem to afflict *all* functional accounts of size. Consider, for instance, the account I mentioned in the previous section, on which a size property is the property of belonging to a given bijection type. This account does not commit to a particular metaphysical theory of what properties are, and the Benacerraf problem, it seems, does not arise for it. Whittle would seem to agree, but he does not consider this to be a problem for his attack on Thesis 1, since he takes the functional accounts immune to Benacerraf's problem to be refuted by the argument discussed in the previous section (see his 2018, fn.15). If my arguments in the previous section are correct, however, then there *is* a genuine problem here for Whittle; for the argument he is relying on fails.

4 Are Cardinalities Size-Like?

Thesis 2 was this:

> The notion of cardinality defined in set theory is at least "size-*like*": it is at least similar to our ordinary notion of size (of sets) and perhaps a natural generalization of the notion of finite size.

Whittle takes cardinalities to be properties of the form

⟨There is a bijection from __ to κ⟩,

where κ is a cardinal number. He makes two related points against Thesis 2 (Whittle 2018, 864):

1. The notion of cardinality is not a generalization of the notion of finite size, since the collection of all cardinalities does not contain the finite sizes. This is allegedly established by the argument from S (discussed in section 3 above), as that argument is supposed to show that finite sizes, unlike finite cardinalities, cannot be understood in terms of bijections.

2. Cardinalities are "just a completely different sort of property from sizes" (Whittle 2018, 864). For, sizes have nothing in particular to do with functions; they are, rather, *intrinsic*: they can be accounted for only in terms of the things that constitute the set, such as its elements, their parts, or their elements. This is supposed to be established by an argument similar to the argument from *S*.

There are, however, two serious problems with this line of argument. First, if my criticism of Whittle's argument from an impossible situation (see section 3 above) is correct, then that argument fails to establish that finite sizes are distinct from finite cardinalities, and it is unclear how a similar argument could establish the supposed intrinsic nature of size properties.

Second, even if the argument from *S* did establish that sizes are intrinsic and that finite sizes are thus distinct from finite cardinalities, it would still *not* follow that sizes are not size-*like*—i.e., not similar to sizes—in any important or interesting ways, or that it is not obvious that they are. For, whether or not sizes are intrinsic, there are several well-known points of similarity between cardinalities and finite sizes that are, arguably, both interesting and important. These include the following:

(i) Finite cardinalities are at least *co-extensive* with finite sizes (Whittle concedes this).
(ii) If a subset *B* of *A* is smaller* than *A* (in the sense of cardinalities), then it is a *proper* subset of *A*; more generally, the pigeonhole principle holds: If a set *B* is smaller* than *A*, then there is no injection from *A* to *B*.
(iii) The following version of Hume's principle (which Whittle accepts for finite sizes) holds: Sets *A* and *B* have the same size* iff there is a bijection between them.
(iv) Assuming the axiom of choice, sizes* (i.e., cardinalities) are well-ordered by the relevant smaller-than* relation.

Given these problems, I submit, Whittle's points against THESIS 2 do not suffice as a refutation of it.

5 Conclusion

Whittle puts forward four arguments against theses 1 and 2; these, as I explained in the introduction, constitute an essential component of his objection

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to Cantor's conception of infinite size. As I hope to have established, however, none of Whittle's arguments against those two theses succeed.*

Ran Lanzet
0000-0003-2475-562X
University of Haifa
lanzetr@gmail.com

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