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‘Unless’ is ‘Or,’ Unless ‘$\neg A$ Unless A’ is Invalid

Roy T. Cook

The proper translation of “unless” into intuitionistic formalisms is examined. After a brief examination of intuitionistic writings on “unless,” and on translation in general, and a close examination of Dummett’s use of “unless” in Elements of Intuitionism (1975b), I argue that the correct intuitionistic translation of “A unless B” is no stronger than “$\neg B \rightarrow A$.” In particular, “unless” is demonstrably weaker than disjunction. I conclude with some observations regarding how this shows that one’s choice of logic is methodologically prior to translation from informal natural language to formal systems.

The topic of this essay is a methodological principle at work within both pedagogical and theoretical contexts—one which is widely accepted, albeit for the most part implicitly and uncritically. The assumption in question is that the translation of informal, natural language claims into one or another formal language is logic neutral. This assumption underwrites our standard logical practices—evidenced both within the classroom and within the peer-reviewed research paper—whereby we first formalize natural language claims into a favored artificial language and only then pronounce judgement on this single, univocal formalization from the perspective of this or that logic.

Here we will see that this methodology is deeply flawed. On the contrary, we must *first* decide which logic (classical, intuitionistic, dialethic, quantum, etc.) is at work, and only then can we provide adequate translations of informal, everyday natural language expressions into whatever formal language is in play. The reason is simple to state, although defending it will require a bit of work: the same natural language expressions should be translated differently with respect to different background logics.

The argument that translation of natural language claims into formal language is neither prior to, nor independent of, our choice of one or more logics as “correct” (or, at least, as the logic currently under consideration) will focus

on a particularly interesting and, in this author's opinion, under-examined example: "unless." As we shall see, the natural language connective "unless" turns out to be a particularly clear case of the phenomenon in question, since the intuitionist should translate claims involving "unless" very differently from the standard rule:

unless = (inclusive) disjunction

commonly taught to students and implicitly accepted in much professional work on logic (including much work on non-classical logic). The remainder of this essay will develop this argument as follows.

First, in section 1, we will look at the standard logical treatment of "unless," where natural language claims of the form "Φ or Ψ" are translated as (or as something *classically* equivalent to) "$\Phi \vee \Psi$," and we will examine the various options available in an intuitionist context, where, for example, "$\Phi \vee \Psi$," "$\neg\Phi \rightarrow \Psi$," and "$\neg\Psi \rightarrow \Phi$" are not equivalent.

In section 2 we will undertake a careful examination of a number of instances of "unless" claims found in *Elements of Intuitionism*, Michael Dummett's classic text on intuitionistic mathematics. As we will see, translating these in terms of disjunction—that is, via application of the rule typically taught to students and uncritically applied by their teachers—produces results that do not accurately capture the content of the original claims. In particular, while the classical logician should (or at least can) translate "unless" claims as disjunctions, the intuitionist should not, since from an intuitionionistic perspective "unless" is weaker than "or."

For the purposes of the remainder of the essay, all that will be needed from section 2 is the relatively weak claim that, intuitionistically at least, claims of the form "Φ unless Ψ" are weaker than claims of the form "Φ or Ψ"—and hence, intuitionists should abandon the "unless-is-or" equation. Interestingly, however, the evidence marshaled in this section supports a stronger claim: the intuitionistically correct translation of natural language claims of the form "Φ unless Ψ" is "$\neg\Psi \rightarrow \Phi$."

In section 3 we will make some additional observations about translation "unless" claims from an intuitionistic perspective, and deal with a few complications raised by the data examined in section 2, including the fact that the translation manual endorsed in that section makes "unless" claims fail to be commutative—that is, "Φ unless Ψ" is not always logically equivalent to "Ψ unless Φ."

Then, in section 4 we will use a toy version of Putnam's argument for quantum logic (1969) to show that the priority of choice of logic to translation in general, and the proper translation of "unless" in particular, is not a trivial or minor matter. In particular, the observations made in previous sections have profound ramifications regarding the shape that arguments for logical revision must take. Perhaps the most striking such consequence is that a particular counterexample to a particular logic—that is, an argument that is valid according to that logic, but which has a true premise and non-true conclusion—can never force us to give up a particular logical law (such as the law that takes us from that premise to that conclusion). Instead, it is always, at least in principle, open to us to argue that the natural language premises and conclusion have been translated incorrectly relative to the standards of the logic in question (which need not be identical to the standards appropriate to the logic with which our opponents wish to replace our own favored system).

Finally, in the concluding section 5 we will tie up some loose ends and note some consequences all of this has for the so-called communication problem: the problem of determining whether or not intuitionists and classical logicians (or any two camps accepting different logics as correct) mean the same thing by logical notions such and "or" and "unless."

1 Translating "Unless"

Consider how "unless" is usually handled in basic logic courses. In such courses, students are often initially confused with regard to how we ought to translate the natural language expression "unless." One common strategy for providing students with some basic insights regarding this translational conundrum is to point out (typically via clear examples) that "unless" seems to obey the following two rules of inference:

$$\frac{\begin{array}{c}\Phi \text{ unless } \Psi \\ \text{Not: } \Phi\end{array}}{\Psi} \qquad\qquad \frac{\begin{array}{c}\Phi \text{ unless } \Psi \\ \text{Not: } \Psi\end{array}}{\Phi}$$

These facts suggest that "Φ unless Ψ" could be plausibly translated as "$\neg\Phi \to \Psi$," or perhaps "$\neg\Psi \to \Phi$" (or perhaps even "$(\neg\Phi \to \Psi) \wedge (\neg\Psi \to \Phi)$" or "$(\neg\Phi \to \Psi) \vee (\neg\Psi \to \Phi)$"). The instructor then typically points out that:

$$\neg\Phi \to \Psi \dashv\vdash_C \Phi \vee \Psi$$
$$\neg\Psi \to \Phi \dashv\vdash_C \Phi \vee \Psi$$

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Hence, the proper translation of "Φ unless Ψ" is "$\Phi \vee \Psi$" (or any of the logical equivalents mentioned above).[1]

Note, however, that all of this depends on the fact that introductory courses on formal logic are typically restricted to instruction on, and from the perspective of, classical logic. Imagine, however, that an intuitionistic logician teaches a course on basic logic (something that happens all the time) and further that she teaches her students intuitionistic logic (H) and teaches it from the perspective of an intuitionist (something that happens far less frequently).

Now, when discussing the proper translation of "unless" claims, even if the intuitionist argued, just as the classical logician did, that both of the argument patterns identified above seem valid (and, as we shall see, there are good reasons for being suspicious of the first argument pattern from an intuitionistic perspective!), she cannot follow her classical counterpart in concluding that this alone shows that "$\Phi \vee \Psi$" is a legitimate translation of "Φ unless Ψ." The reason is simple: The classical logician uses the *classical* logical equivalence of these more complex formulations and "$\Phi \vee \Psi$" to argue that the latter is the preferred, simplest formalization of the natural language expression "unless." For the intuitionist, however, "$\Phi \vee \Psi$" and "$(\neg\Phi \to \Psi) \wedge (\neg\Psi \to \Phi)$" are not equivalent. Moreover, each formula in the following diagram is classical logically equivalent to all of the others, but no two are intuitionistically equivalent (transitive closure of the arrows indicates entailment):[2]

1 Arguably, there is a stronger, *exclusive* reading of "unless"—that is, a reading of "Φ unless Ψ" that entails "not both Φ and Ψ"—that occurs in sentences such as:

> You will get soup unless you get salad.

This reading of unless also has multiple possible, non-equivalent translations for the intuitionist. We will leave construction and consideration of such translation manuals to the energetic reader.

2 Note that we need not restrict our attention to this handful of simple translations. There are many other interesting, disjunction-like operators definable within intuitionistic logic. Interesting examples include *pseudo-disjunction*:

$$\Phi \dot{\vee} \Psi =_{\mathrm{df}} ((\Phi \to \Psi) \to \Psi) \wedge ((\Psi \to \Phi) \to \Phi)$$

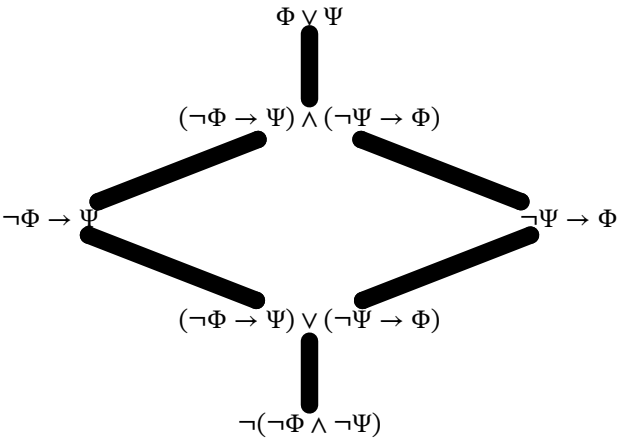
Church disjunction:

$$\Phi \ddot{\vee} \Psi =_{\mathrm{df}} (\Phi \to \Psi) \to ((\Psi \to \Phi) \to \Phi)$$

and *Cornish disjunction*:

$$\Phi \star \Psi =_{\mathrm{df}} (((\Phi \to \Psi) \to \Psi) \to \Phi) \to \Phi$$

These are examined in detail in Humberstone (2011)—the absolutely definitive and authoritative study of propositional connectives in classical and non-classical logics—on pages 555, 235, and 235 respectively.



Thus, there are (at least) six different rules that the intuitionist could adopt for translating "unless":[3]

1. Φ unless $\Psi = \Phi \vee \Psi$

3 Note that only rules [1] through [3] validate the informal inference:

$$\frac{\begin{array}{c}\Phi \text{ unless } \Psi \\ \text{Not: } \Phi\end{array}}{\Psi}$$

and only rules [1], [2], and [4] validate:

$$\frac{\begin{array}{c}\Phi \text{ unless } \Psi \\ \text{Not: } \Psi\end{array}}{\Phi}$$

All six rules validate variants of these rules where the conclusions are replaced by their double negations, of course. Since discussions of translation within logic courses and texts that focus on classical logic elide the difference between $\neg\neg\Phi$ and Φ (if, as the intuitionist claims, there is such a difference), then considering all of these possible translations seems wise, and is, at any rate, harmless even if in the end we accept one or both of these rules as valid on the intuitionistic understanding of "unless."

Here and below, we will speak of translation rule [1] being the "strongest" rule (and rule [6] being the "weakest" rule), as shorthand for the claim that rule [1] outputs the (intuitionistically) strongest translation of "unless" claims (and rule [6] outputs the weakest such translation) with the ordering understood as the partial ordering corresponding to our diagram.

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2. Φ unless $\Psi = (\neg\Phi \rightarrow \Psi) \wedge (\neg\Psi \rightarrow \Phi)$
3. Φ unless $\Psi = \neg\Phi \rightarrow \Psi$
4. Φ unless $\Psi = \neg\Psi \rightarrow \Phi$
5. Φ unless $\Psi = (\neg\Phi \rightarrow \Psi) \vee (\neg\Psi \rightarrow \Phi)$
6. Φ unless $\Psi = \neg(\neg\Phi \wedge \neg\Psi)$

So how should an intuitionist translate "unless"? At the outset, it is worth pointing out one fact that seems like *prima facie* evidence against the claim that "Φ unless Ψ" should be translated as a disjunction: the fact that the "un-" in "unless" seems to encode a negation. Further, the "un-" seems to attach to "Ψ" in particular (as is evidenced by the slightly more pretentious, but presumably equivalent "Unless Ψ, Φ").[4] This suggests (but far from entails) that one of the other, weaker (negation-involving) formulas in the diagram above (i.e. one of rules [2] through [6]) is the best translation of the natural language expression "unless" into intuitionistic logic, and also (but perhaps more weakly) suggests that translation rule [4] is the correct intuitionistic translation of "unless."

There is, of course, another, rather simple way to obtain data relevant to determining the proper translation of "unless": we can just ask intuitionists. In a rare moment of empirical curiosity, and with this in mind, I asked Neil Tennant (via email) how he understood "unless." It turns out that he prefers the "exclusive" reading (see footnote 1), and provided (rules equivalent to) the following introduction rule:[5]

If: $\Delta, \Phi, \Psi \vdash \bot$; $\Delta, \neg\Phi \vdash \Psi$; and $\Delta, \neg\Psi \vdash \Phi$
Then: $\Delta \vdash \Phi$ unless Ψ

and elimination rules:

If: $\Delta_1 \vdash \Phi$; and $\Delta_2 \vdash \Psi$
Then: Δ_1, Δ_2, Φ unless $\Psi \vdash \bot$

4 This argument is analogous, perhaps, to the claim that the "if" in "Ψ, if Φ" attaches, in some sense, to the "Φ," which in turn helps to make vivid the equivalence between this claim and "If Φ then Ψ." Thanks to a referee for pointing this out.

5 A full and proper analysis of Tennant's response would require a bit more subtlety, since his Core Logic involves relevance constraints (e.g. transitivity fails, etc.)—see Tennant (2017). We set these complications aside here, however.

Thanks are of course owed to Tennant for permission to share the upshot of this correspondence.

If: $\Delta, \Phi \vdash \bot$
Then: Δ, Φ unless $\Psi \vdash \Psi$

If: $\Delta, \Psi \vdash \bot$
Then: Δ, Φ unless $\Psi \vdash \Phi$

Extrapolating analogues of these rules for a non-exclusive reading of "unless" is straightforward:

If: $\Delta, \neg\Phi \vdash \Psi$; and $\Delta, \neg\Psi \vdash \Phi$
Then: $\Delta \vdash \Phi$ unless Ψ

If: $\Delta, \Phi \vdash \bot$
Then: Δ, Φ unless $\Psi \vdash \Psi$

If: $\Delta, \Psi \vdash \bot$
Then: Δ, Φ unless $\Psi \vdash \Phi$

These rules clearly correspond to translation rule [2] above, where "Φ unless Ψ" is translated as "$(\neg\Phi \rightarrow \Psi) \wedge (\neg\Psi \rightarrow \Phi)$." Thus, Tennant agrees with what will be one of the main conclusions of this paper: that translating "unless" as (or as equivalent to) "or" is incorrect.[6]

Perhaps the best way to determine how an intuitionist should translate "unless"—better even than asking them directly, given the unreliability of intuitions regarding logical form (and with apologies to Tennant!)—is to study the inferential patterns used by intuitionists when reasoning using "unless."[7] Despite the fact that intuitionists are, sadly, few in number in comparison to their classical opponents, an exhaustive, scientifically compelling linguistic survey of most or all of their publications and pronouncements containing the expression "unless" is far beyond the scope of this essay (and the skills of its author). Thus, I will instead just present close examinations of a few striking and suggestive examples.

6 As we shall see, however, he disagrees with regard to what the *correct* translation is.

7 The point is not that our intuitions about logical form are somehow inherently suspect or are not legitimate data—on the contrary! The point, rather, is that in cases where our armchair, *a priori* philosophical intuitions about logical form conflict with the data obtained by empirically observing how the expressions are actually used (and, as we shall see, Tennant's intuitions and the data presented in the next section are in just such conflict), it seems reasonable to privilege the linguistic data over the intuitions. And, in the interest of full disclosure (and also so Tennant doesn't feel so alone!), my own intuitions agreed with his prior to looking at the data.

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Presumably, we can find no better source for such examples than Michael Dummett's *Elements of Intuitionism* (1977). We will carry out such an examination of *Elements of Intuitionism* in the next section, where we shall see that there is a good bit of evidence in favor of translation rule [4] (and hence against [1]) as the correct intuitionistic translation of "unless"—evidence that is obtained by examining how intuitionists actually use (or, at least, how Dummett actually uses) "unless."

Before moving on, however, there is a complication that we need to deal with. It is well known that, even from a purely classical perspective, translating "unless" as "or" only works in *positive* contexts. In other words, when "unless" occurs in negative contexts, it appears to mean something different. This point is used by Higginbotham (1986), for example, to argue that "unless" is not compositional, since its meaning, and truth conditions, depend on the logical contexts within which it is embedded. Interestingly, Dummett uses "unless" in this sense at least once in *Elements of Intuitionism*:

> No account of the intuitionistic rejection of the law of excluded middle is adequate, therefore, *unless* it is based on the intuitionistic rejection of the platonistic notion of mathematical truth as obtaining independent of our capacity to give a proof. (1977, 12, emphasis added)

Even on a classical understanding of this claim, translating "unless" as "or" (or any of its logical equivalents discussed above) is inadequate, since doing so would entail that the quotation above is equivalent to:[8]

> It is not the case that there is an x such that either x is an adequate account of the intuitionistic rejection of the law of excluded middle or x is based on the intuitionistic rejection of the platonic notion of mathematical truth as obtaining independent of our capacity to give a proof.

8 Note that applying the exclusive reading of "unless" (i.e. "unless" as equivalent to exclusive disjunction) to this passage

> It is not the case that there is an x such that x is an adequate account of the intuitionistic rejection of the law of excluded middle if and only if x is not based on the intuitionistic rejection of the platonic notion of mathematical truth as obtaining independent of our capacity to give a proof

works no better.

This has the following logical form:

$$\neg(\exists x)(F(x) \vee G(x))$$

Higginbotham argues that occurrences of "unless" embedded in such negative contexts should be translated instead as "and not," resulting in something like[9]

> It is not the case that there is an x such that x is an adequate account of the intuitionistic rejection of the law of excluded middle and it is not the case that x is based on the intuitionistic rejection of the platonic notion of mathematical truth as obtaining independent of our capacity to give a proof

which has the following logical form:

$$\neg(\exists x)(F(x) \wedge \neg G(x))$$

This suggestion seems to capture at least the classical content of Dummett's claim relatively well—that is, it adequately captures how the sentence should be translated into the language of classical logic if the sentence had been uttered by a classical logician. Of course, given the occurrence of both existential quantification and negation—two notions that are central to the disagreement between classical and intuitionistic accounts of logic—in this translation, there might well be reasons to think that the intuitionistic translation of this passage should be different, similar to the reasons we shall see in the next section for objecting to the intuitionistic translation of "unless" as "or" in positive contexts.

For the sake of keeping this essay relatively short(ish) and snappy(ish), however, we will set aside the issue of translating "unless" when it occurs in the scope of negated quantifiers. The interested reader is encouraged to carry out their own textual analysis, similar to the one carried out for occurrences of "unless" in positive contexts, in order to determine if the intuitionist should adopt the same "and not" rule as the classical logician, or some intuitionistically non-equivalent (but presumably classically equivalent) formulation.

9 There is, of course, a significant literature in logic and linguistics arguing for various other ways of handling "unless" in negative contexts, including accounts that aim for a uniform approach that salvages compositionality. Since we are setting aside negative occurrences of "unless" here, we need not survey such accounts (interesting though they might be!)

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2 "Unless" in *Elements of Intuitionism*

Let's now begin our examination of Dummett's use of "unless" in *Elements of Intuitionism*. We will not attempt to consider every occurrence of this expression in Dummett's book (we will include a footnote listing some additional examples, and explaining why they were not examined in detail here, near the end of this section). Instead, we will look at enough cases to:

- Demonstrate that translating intuitionistic utterances of "unless" as "or" is too strong—that is, we should reject translation rule [1] above in favor of something weaker (such as any of [2] through [6]).
- Construct a significant body of evidence in favor of translation rule [4] as the correct rule for translating informal "unless" claims into intuitionistic formal languages (i.e. "Φ unless Ψ" should be translated as "$\neg\Psi \rightarrow \Phi$").

Of course, the latter claim (that translation rule [4] is the correct rule) entails the former claim (that translation rule [1] is incorrect). But keeping these two claims separate in this way is useful for two reasons. First, I think it likely that many readers will find my arguments against rule [1] to be more definitive than my arguments in favor of rule [4]. As we've already noted, Tennant agrees with me about [1] being incorrect, but disagrees regarding rule [4] being the correct rule. Equally important, however, is the second reason for keeping these two claims separate: regardless of the ultimate fate of the latter, stronger claim, the incorrectness of translation rule [1] is all that is needed for the further arguments regarding logical revision that will be presented in section 4 below.

We will work through the first example in full and gory detail, and then work through additional examples somewhat more quickly and loosely. For our first such example, consider the following passage:

> A quasi-completeness proof of this kind can plainly be given only for a fragment of predicate logic within which the intuitionistically and classically provable formulas coincide (and not, as Kreisel points out, for every such fragment). As for the general case, it is evident from Theorem 5.37 that, *unless* we are prepared to accept schema (11) for primitive recursive predicates, we have no hope of proving even the quasi-completeness of any formalization of intuitionistic logic for which the extended Hauptsatz,

> which is a version of Herbrand's Theorem, holds. (Dummett 1977, 182)

In order to asses this occurrence of "unless," we need some of the mathematical background.

A logical system is *quasi-complete* if and only if every unprovable formula fails to hold on every internal interpretation (which is weaker than the requirement that there is a particular internal interpretation on which it does not hold). Schema (11) is:

$$(\forall \vec{u})(A(\vec{u}) \vee \neg A(\vec{u})) \wedge \forall \alpha \neg\neg \exists n A(\bar{\alpha}(n)) \rightarrow \neg\neg \forall \alpha \exists n A(\bar{\alpha}(n))$$

and the relevant portion of Theorem 5.37 states that, if HPC (intuitionistic predicate logic) is internally quasi-complete then all instances of Schema (11) hold where $A(\vec{x})$ is primitive recursive.[10]

So, with this in mind, how should we translate Dummett's claim that,

> [...] *unless* we are prepared to accept schema (11) for primitive recursive predicates, we have no hope of proving even the quasi-completeness of any formalization of intuitionistic logic for which the extended Hauptsatz, which is a version of Herbrand's Theorem, holds". (1977, 182)

Let's simplify Dummett's claim a bit, and instead consider the (slightly less poetic, but more precise) statement

> We are unable to prove the quasi-completeness of any formalization of HPC for which the Hauptsatz holds, unless we have reason to accept schema (11) for primitive recursive $A(\vec{x})$

and adopt the following translation manual:

> A = We are able to prove the quasi-completeness of some formalization of intuitionistic logic for which the Hauptzatz holds.
>
> B = We have reason to accept schema (11) for primitive recursive $A(\vec{x})$.

10 It is worth noting that Dummett also proves that, if Schema (11) holds for *all* $A(\vec{x})$ (primitive recursive or not), then ICP is internally quasi-complete for single formulas.

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If translation rule [1], where "unless" is just disjunction, were correct, then we should formalize Dummett's claim as:

$$\neg A \vee B$$

Translating this back into natural language, this would entail that Dummett's claim is equivalent to the following:

> Either it is not the case that we are able to prove the quasi-completeness of any formalization of intuitionistic logic for which the extended Hauptsatz holds, or we have reasons to accept schema (11) for primitive recursive predicates.

Now, an intuitionist typically (and Dummett definitely) treats disjunction as determinate, in the sense that "$\Phi \vee \Psi$" is taken to be equivalent to something like:

> Φ is definitely the case, or Ψ is definitely the case.

or:

> Φ is the case, or Ψ is the case (and we can determine which).

Given this, however, the result of applying translation rule [1] to Dummett's natural language claim is immensely implausible. Earlier in the same chapter Dummett writes that:

> Unfortunately, there is no particular reason for supposing schema (11) to be intuitionistically valid; it can again be shown to be underivable in the usual systems of intuitionistic analysis, although there is not the same positive reason to suppose it invalid as there was in the case of (9). (1977, 176)

This quotation concerns, of course, schema (11) in full generality, rather than restricted to primitive recursive predicates, but the open status of (11) restricted to primitive recursive predicates is clearly expressed in a paper of Kreisel's upon which much of Dummett's discussion depends:[11]

11 "(3)" is Kreisel's label for (a principle shown by Dummett to be equivalent to) (11) restricted to primitive recursive predicates, and "weak completeness" is an alternative term for "quasi-completeness."

> [...] (3) is not so implausible, and may be provable on the basis of as yet undiscovered axioms which hold for the intended interpretation (but not for the realizability interpretations). So the problem whether HPC is weakly complete is still open. (1962, 4)

Thus, it is neither the case that we definitely know (can prove) schema (11) restricted to primitive recursive predicates, nor that we can definitely refute (11) so restricted. And clearly such indecision applies to the claim about proving quasi-completeness as well: if we (i.e. Dummett, when writing the text) knew either that the internal quansi-completeness of ICP could be proven, or that it could be refuted, then surely he would have included such a proof (or at least a report of such a proof) in a chapter on completeness proofs for intuitionistic systems (or see the final sentence in the Kreisel quotation above).

But what about applying translation rules [2] through [6]? Which of the remaining translations of Dummett's "unless" claim into the language of intuitionistic logic are plausible, and which are not? If we apply our translation manual and rule [3], we obtain:

$$\neg\neg A \rightarrow B$$

which then translates back into informal prose as:

> If it is not the case that it is not the case that we have reasons to accept schema (11) for primitive recursive predicates, then we are able to prove the quasi-completeness of some formalization of intuitionistic logic for which the extended Hauptsatz holds.

This claim does not follow from Theorem 5.37 as stated, however. Theorem 5.37, as stated, amounts to:

$$A \rightarrow B$$

We can, of course, apply contraposition twice to obtain:

$$\neg\neg A \rightarrow \neg\neg B$$

which then translates back to something like:

> If it is not the case that it is not the case that we are able to prove the quasi-completeness of any formalization of intuitionistic logic for which the extended Hauptsatz holds, then it is not the case that

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> it is not the case that we have reasons to accept schema (11) for primitive recursive predicates.

This does follow from Theorem 5.37. But this claim is strictly speaking weaker than the result of applying translation rule [3] (i.e. it is intuitionistically entailed by, but does not intuitionistically entail, the translation that results from applying rule [3]).

Given that Dummett asserts that the "unless" claim in question is *evident* from Theorem 5.37, this strongly suggests that translation rule [3] (and hence also against the stronger rule [2]) does not deliver the correct translation of "unless," since the result of applying this translation does not, contrary to Dummett's claim, actually follow from Theorem 5.37.[12]

Translation rule [4] fares better, however. Applying rule [4], we obtain:

$$\neg B \rightarrow \neg A$$

which translates back into prose as:

> If it is not the case that we have reasons to accept schema (11) for primitive recursive predicates, then it is not the case that we are able to prove the quasi-completeness of any formalization of intuitionistic logic for which the extended Hauptsatz holds.

This just is the contrapositive of Theorem 5.37—hence, it is clearly evident to anyone who considers that theorem and is aware of the intuitionistic validity of contraposition.[13] In addition, this translation is strictly weaker than the translation obtained via application of rule [3] (i.e. the latter intuitionistically entails the former).[14] Hence this seems like a perfectly adequate (and, given

12 It is important to note that the argument does not depend on the result of applying translation rule [3] being false (or failing to be true, etc.) The point is that the result of applying this translation rule does not result in a translation whose truth follows immediately from the theorem in question.

13 The technical term "contraposition" can refer to a number of different (classically equivalent) rules. Here we mean:

$$\Phi \rightarrow \Psi \vDash \neg\Psi \rightarrow \neg\Phi$$

and not, for example:

$$\neg\Phi \rightarrow \Psi \vDash \neg\Psi \rightarrow \Phi$$

The latter is, of course, not intuitionistically valid. Thanks are owed to an anonymous referee for suggesting this clarification.

14 Note that it is not the case in general that the translation delivered by rule [3] entails the translation delivered by rule [4]. Hence, the fact that this entailment holds with regard to the results

the options we are considering, the *strongest* adequate) way of translating this "unless" claim into the language of intuitionistic logic.

This example shows that, if we are looking for a uniform rule for translating informal "unless" claims into the formal language of the intuitionist—one that respects their actual usage of "unless"—then translating "unless" as disjunction is unacceptable, and in addition, the strongest possible such rule (at least, amongst the relatively simple rules we are considering here) that applies to all intuitionistic uses of "unless" is rule [4].[15] In order to see that this is not an isolated case, we will look at a few more examples.

Dummett writes the following in the preface to the first edition:

> Intuitionistic mathematics cannot be justified by its purely 'mathematical interest': one subject-matter may differ from another according to the degree of mathematical interest which they have; but a set of principles of mathematical reasoning, diverging in both directions from those usually accepted, is devoid of interest *unless* there is some way of understanding mathematical statements in accordance with which those principles are justified and other principles are not. (1977, ix, emphasis added)

Adopting the disjunctive rule [1], the claim in question becomes:

> For every set of principles diverging from those usually accepted, either it is devoid of interest or there is some way of understanding mathematics in accordance with which those principles are justified and others are not.

Again, as in our first example, this seems (on the intuitionistic understanding of "or") too strong: surely Dummett is not claiming that we have a *method* for determining, of each such system that diverges from classical mathematics, whether it is devoid of interest or it is justified in the way he describes.

Translation rules [3] and [4] both fare better with this example. On translation rule [3] the passage above turns out to be equivalent to:

---

of applying these rules to most of the actual instances of "unless" that occur in *Elements of Intuitionism* is an interesting fact, which we shall return to in the next section.

15 Of course, one could perhaps argue that Dummet is speaking loosely here, or is uncharacteristically misusing the expression, or… [fill in one's favorite *ad hoc* explanation for why this example is atypical]. Presumably, if one allows this strategy, then one can just cherry-pick whatever examples fit one's preconceptions about the intuitionistic meaning of "unless"– a strategy that seems neither methodologically respectable nor likely to be fruitful.

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> For every set of principles diverging from those usually accepted, if it is not devoid of interest then there is some way of understanding mathematics in accordance with which those principles are justified and others are not.

And on translation rule [4] it is equivalent to:

> For every set of principles diverging from those usually accepted, if there is no way of understanding mathematics in accordance with which those principles are justified and others are not, then it is devoid of interest.

Note, however, that in this particular example (and like the previous example), the result of applying translation rule [3] in this case is *logically* stronger than the result of applying rule [4] due to the presence of an embedded negation. Let us adopt the following translation manual (somewhat loosely put):

> $A(x) =$ Mathematical principles x are of some interest.
>
> $B(x, y) = y$ is a way of understanding mathematical principles x.

Hence, "x is devoid of interest" is "$\neg A(x)$" then the result of applying translation rule [3] is:

$$(\forall x)(\neg\neg A(x) \rightarrow (\exists y)(B(x, y))$$

and the result of applying translation rule [4] is:

$$(\forall x)(\neg(\exists y)(B(x, y)) \rightarrow \neg A(x))$$

Note that the latter formula is (intuitionistically) a logical consequence of the former.

The translation we obtain by applying rule [3] says something like: If it isn't the case that a particular system is devoid of interest, then *there is* (i.e. there is a method by which we can find) a way of understanding its principles such that those principles are justified and others are not. But Dummett's original claim does not seem to imply that there is, for each such system that is not devoid of interest, a corresponding way to find a suitable interpretation of that system. If this is right, then we again have evidence that not only is rule [1] incorrect, but rule [3] (and hence rule [2]) is incorrect as well, since it produces translations that are (intuitionistically) stronger than the informal

claims being translated. The translation obtained by applying translation rule [4], however, seems nicely in line with what Dummett actually seems to be saying, evaluated along intuitionistic lines.

Let's look at another example. In his discussion of the failure of the least number principle, Dummett writes that:

> We should note, however, that the least number principle:
>
> $$\exists x A(x) \rightarrow \exists x(A(x) \wedge \forall y_{y<x} \neg A(y))$$
>
> is *not* intuitionistically valid: *unless* $A(x)$ happens to be decidable, the fact that we can find a definite number n of which we can prove that it satisfies $A(x)$ is no guarantee that we can find any number m satisfying $A(x)$ of which we can show that no smaller number satisfies it. (1977, 23, emphasis added)

Applying translation rule [1] (and reading a bit into what kind of guarantee Dummett has in mind), this claim is equivalent to:

> For any predicate $A(x)$, either $A(x)$ is decidable, or the fact that there is an x such that $A(x)$ is (on its own) no guarantee that there is a least x such that $A(x)$.

This, again, seems to be too strong, since it implies that, for any predicate $A(x)$, we have some method for determining either that $A(x)$ is decidable or that there is no guarantee that the least number principle holds for $A(x)$.[16]

Applying translation rule [3], the quotation in question turns out to be equivalent to:

> For any predicate $A(x)$, if it is not the case that:
> the existence of an x such that $A(x)$ is (on its own) no guarantee that there is a least x such that $A(x)$,
> then $A(x)$ is decidable.

There doesn't seem to be any obvious reason to think that this claim is even true: the fact that we can refute the claim that there is no guarantee of the relevant sort seems to fall far short of being able to determine that $A(x)$ is decidable.

16 Another way of putting the worry is this: The result of applying rule [1] to this example seems to imply that whether or not $A(x)$ is decidable, for arbitrary (arithmetical) $A(x)$, is itself decidable.

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Translation rule [4], however, makes the original quotation equivalent to something like:

> For any predicate $A(x)$, if $A(x)$ is not decidable, then the existence of an x such that $A(x)$ is (on its own) no guarantee that there is a least x such that $A(x)$.

This, unlike the result of applying rule [1] or rule [3], seems to capture exactly what Dummett's original "unless" claim was meant to express. In addition, note that, once again, the result of applying rule [3] entails the result of applying rule [4].

Here's another example. Dummett writes:

> That is not to claim that an understanding of any sentence could exist on its own, without a knowledge of any of the rest of the language: every sentence is composed of words or signs which could not be understood *unless* it were known how to use them in at least some other sentences. (1977, 255, emphasis added)

If we adopt translation rule [1], then the sentence at the end of this passage is equivalent to:

> Every sentence is composed of words such that either they cannot be understood or their use in at least some other sentences is known.

Given the intuitionistic understanding of "or," this is clearly too strong, since it implies that, for every sentence, we can decide whether we understand the words contained in it. Translation rule [3] gives us:

> Every sentence is composed of words such that, if it is not the case that they cannot be understood, then their use in at least some other sentences is known.

And translation rule [4] gives us:

> Every sentence is composed of words such that, if it is not the case that their use in at least some other sentences is known, then they cannot be understood.

Note that here (as in all of our other examples), the presence of an embedded negation (along with equating "*x* cannot be understood" with "it is not the case that *x* can be understood") makes it the case that the result of applying translation rule [3] is strictly stronger than the result of applying translation rule [4]—that is, the former logically entails the latter.

Since this passage, unlike our earlier examples, is more informal, our results will be a bit less definitive. Nevertheless, the translation rule [3] result seems odd (to the author at least)—the strange double negation construction in the antecedent does not seem to be part of the content of Dummett's informal claim. The result of applying translation rule [4], however, once again seems to capture exactly what Dummett means (and, if one disagrees with the claim that the result of applying translation rule [4] better captures Dummett's meaning than the result of applying translation rule [3], this does not affect the claim that applying translation rule [1] is just incorrect!)

Let us look at one final example. Dummett writes that:

> The upshot of our review of this second approach is that the status of mathematical objects, as existing independently of us or as the products of our own thought, is irrelevant to whether a classical interpretation of the logical constants is admissible or whether they can be interpreted only in the intuitionistic sense, *unless* the thesis that such objects are the products of our thought it understood in the most radical manner possible, namely as entailing that even primitive predicates (and ones compounded from these by the sentential operators and quantification over a finite domain) are true of them only when we have expressly recognized them to be. To what extent such a radical anti-realism with respect to the objects of mathematics is defensible, and to what extent it is compatible with realism about the contents of the physical universe, are questions left to the reader to think through. (1977, 269, emphasis added)

Applying translation rule [1] implies that the above claim is equivalent to something like:

> Either the status of mathematical objects, as existing independently of us or as the products of our own thought, is irrelevant to whether a classical interpretation of the logical constants is admissible or whether they can be interpreted only in the intuitionistic sense, or

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> the thesis that such objects are the products of our thought must be understood in the most radical manner possible, namely as entailing that even primitive predicates (and ones compounded from these by the sentential operators and quantification over a finite domain) are true of them only when we have expressly recognized them to be.

Again, given the particularly strong reading that intuitionists attach to "or," this just seems too strong: Dummett does not seem to be claiming here that we can tell which of the two subclaims holds—in fact, the sentence that follows immediately after the "unless" claim in the original passage seems to suggest just the opposite (whatever we might suspect Dummett's actual views on these matters are).

Applying translation rule [3], we obtain something like:

> If it is not the case that the status of mathematical objects, as existing independently of us or as the products of our own thought, is irrelevant to whether a classical interpretation of the logical constants is admissible or whether they can be interpreted only in the intuitionistic sense, then the thesis that such objects are the products of our thought must be understood in the most radical manner possible, namely as entailing that even primitive predicates (and ones compounded from these by the sentential operators and quantification over a finite domain) are true of them only when we have expressly recognized them to be.

And applying translation rule [4], we obtain something like:

> If it is not the case that the thesis that such objects are the products of our thought is understood in the most radical manner possible, namely as entailing that even primitive predicates (and ones compounded from these by the sentential operators and quantification over a finite domain) are true of them only when we have expressly recognized them to be, then the status of mathematical objects, as existing independently of us or as the products of our own thought, is irrelevant to whether a classical interpretation of the logical constants is admissible or whether they can be interpreted only in the intuitionistic sense.

Again, the translation obtained by applying rule [4] seems more natural than the translation obtained via applying rule [3], although, unlike the earlier cases, I see no definitive reasons for thinking that the result of applying translation rule [3] (or translation rule [2]) in this case gives the *wrong* result.[17]

This concludes our discussion of examples that show that we should reject translation rule [1] and that, in addition, we should favor rule [4] over the rest.[18] Before moving on, however, it is worth noting that there are instances of "unless" in *Elements of Intuitionism* that could, in isolation, be read as (or as equivalent to) disjunctions. For example, in presenting the proof that there are infinitely many logically non-equivalent formulas containing a single sentence letter p, Dummett writes that:[19]

> There are denumerably many non-equivalent formulas with a single sentence-letter p, which form a highly memorable structure.

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17 Although things are a bit more complicated here, the result of applying rule [3] again seems to entail the result of applying rule [4]. The former has something like:

> If (not: not: Relevant(status of math, interpretation admissible)) then (Understood Radically(thesis that math is thought))

as its logical form, while the latter has something like:

> If (not: Understood Radically(thesis that math is thought)) then (not: Relevant(status of math, interpretation admissible))

18 There are at least two other instances of "unless" in Dummett (1977) that we could consider. The first is on page 299, and the second is on page 305, and both are embedded in complicated bits of reasoning concerning choice sequences. Thus, we have left out explicit discussion of them here, since clarifying the relevant mathematics would take us too far afield and kill too many trees. The reader is encouraged, however, to consider these additional examples, and verify that in both cases translation rule [1] is inappropriate.

19 For an informal example where translation rule [1] seems compatible with the facts, consider:

> If there is a flaw at the heart of classical mathematics, then, even if the intuitionistic reconstruction of mathematics is not correct in every detail, something along those general lines must be right, *unless*, as is surely unthinkable, all but the most elementary parts of arithmetic are delusory. (1977, 250, emphasis added)

There is at least some reason to think, however, that the relative naturalness of reading this passage as an instance of disjunction (in comparison to the cases canvassed above, which cannot be so read) is that the passage is really an explicit assertion of "Φ unless Ψ" and, in addition, an implicit assertion of "it is not the case that Ψ" (indicated by "as is surely unthinkable"). Hence, if we apply translation rule [4], we obtain "$\neg\Psi \rightarrow \Phi$" which, combined with "$\neg\Psi$," entails "$\Phi$," which in turn entails "$\Phi \vee \Psi$."

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> Let us set $P_0 = p \wedge \neg p$, $P_1 = p$, $P_3 = \neg\neg p$, $P_4 = p \vee \neg p$, $P_5 = \neg\neg p \rightarrow p$, $P_6 = \neg p \vee \neg\neg p$, and, for $n > 2$, $P_{2n+2} = P_{2n-1} \rightarrow P_{2n-2}$ and $P_{2n+2} = P_{2n-3} \vee P_{2n-1}$. Then none of the formulas P_n is intuitionistially valid, and every formula with the single sentence-letter p is equivalent to P_n for some n, *unless* it is intuitionistically valid, in which case it is of course equivalent to $p \rightarrow p$; [...]. (1977, 21, emphasis added)

Given the fact that intuitionistic propositional logic is decidable, and given the fact that the construction he sketches here provides a method for identifying, for any formula containing only the single sentence letter p, the particular P_n that is its equivalent (for any propositional formula Φ in p, merely apply the decision procedure to $\Phi \leftrightarrow P_0$, then to $\Phi \leftrightarrow P_1$, then to $\Phi \leftrightarrow P_2$, and so on, until one find the true equivalence), the following claim is, in fact, intuitionistically justified:

> For any formula with the single sentence letter p, either it is equivalent to P_n for some n, or it is intuitionistically valid.

The mathematical and logical facts being consistent with this stronger reading no more implies that we should understand this instance of "unless" as a disjunction, any more than a day where the weather alternates between rain and snow implies that we should understand my assertion of:

> It will rain unless it snows.

as equivalent to the conjunction:

> It will rain and it will snow.

Thus, this example in no way throws doubt on the claim that translation rule [1] is too strong.[20]

20 An anonymous referee pointed out the following from Brouwer's "Points and Spaces," which was originally published in English:

> [...] the wording of a mathematical theorem has no sense unless it indicates the construction either of an actual mathematical entity or an incompatibility (e.g. the identity of the empty two-ity with an empty unity) out of some constructional condition imposed on a hypothetical mathematical system. (1954, 3)

If we apply rule [1], we obtain something like the following:

3 Some Additional Observations

Before discussing the upshot that the observations made in the previous section have for debates about logic and logical revision, there are two additional issues regarding the proper translation of "unless" that should be dealt with.

First, we should be careful regarding what, exactly, we have shown with regard to translation rule [4]. The sort of evidence presented in the previous section is merely evidence that the *strongest* rule compatible with the evidence in question is rule [4]. Of course, there are presumably good *prima facie* reasons, when translating a natural language expression into a formal language, for taking strongest translation compatible with the evidence (reasons of charity, assumptions of maximal infomativeness, etc.). But these consideration will of course compete with other (themselves *prima facie*) considerations.

One such consideration is worth mentioning here: the strong intuition that "unless" is commutative—that is, the strong intuition that whatever translation rule we adopt, it should support the following equivalence (where L is *whatever* logic we are using):

$$\Phi \text{ unless } \Psi \dashv\vdash_{\mathsf{L}} \Psi \text{ unless } \Phi$$

If we adopt translation rule [4] however, then one obvious result of this is that "unless" claims, in the mouths of intuitionists, will not, in general, be commutative. A nice example of this is given by considering various claims

---

> Either the wording of a mathematical theorem has no sense or it indicates the construction either of an actual mathematical entity or an incompatibility [...].

This seems stronger than what Brouwer intends here (since it entails that whether or not a theorem is sense-less or indicates an appropriate construction is decidable). Paraphrasing loosely along the lines of rule [3] gives us:

> If the wording of a mathematical theorem fails to have no sense then it indicates the construction either of [...] or [...].

This does not seem obviously too strong, but the presence of the awkward double-negation (which is absent in the original, "unless"-containing sentence) seems odd. A similarly loose application of rule [4] provides:

> If the wording of a mathematical theorem does not indicate the construction either of [...] or [...], then it has no sense.

This seems (to the author, at least) to capture exactly what Brouwer had in mind.

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that are classically equivalent to excluded middle but are expressed in terms of "unless," such as:

$$\Phi \text{ unless } \neg\Phi$$

$$\neg\Phi \text{ unless } \Phi$$

Of course, for the classical logician, each of these will be equivalent to excluded middle (and hence a logical truth) regardless of which translation rule they adopt. But, if translation rule [4] is correct, then for the intuitionist these amount, respectively, to:

$$\neg\neg\Phi \rightarrow \Phi$$

$$\neg\Phi \rightarrow \neg\Phi$$

The first is classically but not intuitionistically valid. The second, however, is an intuitionist logical truth. Hence, they are far from being equivalent.

I myself do not have this intuition regarding the commutativity of (intuitionistic) "unless"—on the contrary, as mentioned at the beginning of this essay, I think the fact that the "un" in "Φ unless Ψ" seems (i) to indicate the presence of a negation, and (ii) to attach to "Ψ" but not to "Φ" to be evidence that "Φ" and "Ψ" are not on a par, so to speak, in "Φ unless Ψ."

Nevertheless, the reader who is convinced (for whatever reasons) that "unless" is commutative should not, given the evidence just presented, insist that this means that we should adopt translation rule [1] or translation rule [2], despite the fact that these rules deliver commutative translations of "unless" claims—after all, the examples discussed above show that applying either of these rules results in a translation that is intuitionistically stronger than the informal natural language claim being translated.

In addition, the commutativity-sympathetic intuitionist cannot adopt rule [4], but then stipulate that "unless" is, contrary to what the translation might suggest, commutative. In other words (if one wants to remain an intuitionist of some sort) one should not adopt rule [4] but then use a logic H^* where H^* is intuitionistic logic H plus the following additional rule of inference:

$$\Phi \text{ unless } \Psi \dashv\vdash_{H^*} \Psi \text{ unless } \Phi$$

The reason is simple: adding this rule to intuitionistic logic (combined with rule [4]) just results in classical logic. Let Φ be any formula in our formal language. Clearly $\vDash_{H^*} \neg\Phi \rightarrow \neg\Phi$. But, given rule [4], this is equivalent to $\vdash_{H^*} \neg\Phi$ unless Φ. By our commutativity rule, this gives us $\vdash_{H^*} \Phi$ unless $\neg\Phi$.

Applying translation rule [4] again gives $\vdash_{H^*} \neg\neg\Phi \to \Phi$. Since Φ was arbitrary, it follows that $H^* = C$.

Instead, if one is absolutely committed to the commutativity of "unless"—even in intuitionistic contexts—then the correct response is to adopt translation rule [5]. On this reading, each of the claims of the form "Φ unless Ψ" discussed should be translated as:

$$(\neg\Phi \to \Psi) \vee (\neg\Psi \to \Phi)$$

Given the intuitionistic strength of disjunctions, it strikes me – intuitively, at least—that such a translation does some violence to the intended meanings of the passages quoted above. Nevertheless, there is an interesting fact that we need to take into account before putting too much weight on this observation.

In every single one of the examples discussed above (other than the final example, which was compatible with translation rule [1]), the "Φ unless Ψ" claim that we were examining was one where the Φ in question was a negated claim:[21]

- We are *unable* to prove the quasi-completeness of any formalization of HPC for which the Hauptsatz holds, unless [...].
- A set of principles of mathematical reasoning is *devoid* of interest unless [...].
- For any predicate $A(x)$, the fact that there is an x such that $A(x)$ is (on its own) *no* guarantee that there is a least x such that $A(x)$ unless [...].
- Every sentence is composed of words or signs which *could not* be understood unless [...].
- The status of mathematical objects, as existing independently of us or as the products of our own thought, is *irrelevant* to whether a classical interpretation of the logical constants is admissible or whether they can be interpreted only in the intuitionistic sense, unless [...].

In other words, each of these examples is really of the form "$\neg\Phi$ unless Ψ." And, although rule [4] and rule [5] do not deliver logically equivalent translations, they do deliver equivalent translations for cases of this sort, where the expression which is not directly after "unless" is a negated expression. In other words, although $\neg\Psi \to \Phi$ is not (intuitionistically) logically equivalent to $(\neg\Phi \to \Psi) \vee (\neg\Psi \to \Phi)$, $\neg\Psi \to \neg\Phi$ is (intuitionistically) logically equivalent

21 This is also true of the examples we did not discuss in detail, in Dummett (1977, 250, 299, 305).

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to $(\neg\neg\Phi \rightarrow \Psi) \vee (\neg\Psi \rightarrow \neg\Phi)$. As a result, translation rule [5] will fare just as well as a translation of any of the examples discussed above as did translation rule [4].

Thus, if the intuitionist believes they have good reasons to retain the commutativity of "unless," then they can adopt rule [5] rather than rule [4]. As I have noted, I don't see good reasons for thinking that intuitinistic uses of "unless" must be commutative, and I find the translations that result from applying rule [5] to the passages examined in the previous section to be overly complicated, and to do a worse job at capturing Dummett's intended meaning, in comparison to the translations delivered by rule [4]. But for now we can set this aside, since none of the points made in the remainder of this essay depend on rule [4] being correct (or even on rules [2] and [3], much less rule [5], being incorrect): all that is required for the discussion of logical revision in the next section is that rule [1] is definitely incorrect, and nothing said here about commutativity affects our argument for that, much weaker, conclusion.

The second issue is this: why assume that there is a single, univocal, correct translation of "unless" into our formal languages in the first place? Throughout this essay we have assumed that there is such a correct translation rule, and we have then compared and contrasted rules [1] through [6] as candidates for this single, correct rule. But this might be a fallacy. After all, from an intuitionistic standpoint, the following claim:

> For any "$\Sigma(\Phi, \Psi)$" in standard propositional logic, if "$\Sigma(\Phi, \Psi)$" is the *correct* translation of the natural language expression "Φ unless Ψ" then:
>
> "$\Sigma(\Phi, \Psi)$" is no stronger than "$\Phi \vee \Psi$,"
>
> and:
>
> "$\Sigma(\Phi, \Psi)$" is no weaker than "$\neg(\neg\Phi \wedge \neg\Psi)$,"

which we quickly accepted at the very beginning of this essay, does not (intuitionistically) entail that:

> There is a"$\Sigma(\Phi, \Psi)$" in the language of propositional logic such that "$\Sigma(\Phi, \Psi)$" is the single, unique *correct* translation of the natural language expression "Φ unless Ψ."

Restricting our attention to the six competing translations rules we have explicitly discussed in this essay, we can formalize the former claim as something

like:

$$(\forall x)(\mathrm{Corr}(x) \rightarrow (x = \mathrm{rule}[1] \vee x = \mathrm{rule}[2] \vee x = \mathrm{rule}[3] \\ \vee x = \mathrm{rule}[4] \vee x = \mathrm{rule}[5] \vee x = \mathrm{rule}[6]))$$

(where "Corr(x)" expresses the claim that x is the correct rule for translating "unless" into our formal language), and we can formalize the latter as:

$$\mathrm{Corr}(\mathrm{rule}[1]) \vee \mathrm{Corr}(\mathrm{rule}[2]) \vee \mathrm{Corr}(\mathrm{rule}[3]) \\ \vee \mathrm{Corr}(\mathrm{rule}[4]) \vee \mathrm{Corr}(\mathrm{rule}[5]) \vee \mathrm{Corr}(\mathrm{rule}[6])$$

The former claim does not intuitionistically entail the latter. In fact, the former claim, plus the additional claim that it is not the case that all six rules fail to be correct—that is:

$$\neg(\neg\mathrm{Corr}(\mathrm{rule}[1]) \wedge \neg\mathrm{Corr}(\mathrm{rule}[2]) \wedge \neg\mathrm{Corr}(\mathrm{rule}[3]) \\ \wedge \neg\mathrm{Corr}(\mathrm{rule}[4]) \wedge \neg\mathrm{Corr}(\mathrm{rule}[5]) \wedge \neg\mathrm{Corr}(\mathrm{rule}[6]))$$

do not jointly entail that one of the six rules must be correct.[22]

To get that conclusion, we need to assume, in addition, that some rule is, in fact, correct—that is, we need to assume:[23]

$$(\exists x)(\mathrm{Corr}(x))$$

But perhaps we should not make this additional, rather substantial assumption. We certainly have not given an argument for this claim here. Perhaps, for example, all we have justification for is the (intuitionistically weaker) claim that it can't be the case that all of rules [1] through [6] fail to be correct. After all, the failure of claims of this form to entail the corresponding disjunctions—that is, the invalidity of the relevant instance of the DeMorgan equivalences—is one of the distinctive features of intuitionistic logic. Maybe there is no single rule that correctly translates all "unless" claims (even when restricting attention to positive contexts), even though every occurrence of "unless" should be translated as no stronger than the result of applying rule [1] (or, given the arguments made above, perhaps rule [2]) and no weaker

22 A sketch of the Kripke model: There are seven worlds $w_0, w_1, w_2, w_3, w_4, w_5, w_6$. The domain of each world is $\{\mathrm{rule}_1, \mathrm{rule}_2, \mathrm{rule}_3, \mathrm{rule}_4, \mathrm{rule}_5, \mathrm{rule}_6\}$. For each n, $0 < n \leq 6$, $R(w_0, w_n)$, and for each n, $0 \leq n \leq 6$, $R(w_n, w_n)$. Corr holds of nothing at w_0, and for each n, $0 < n \leq 6$, $\mathrm{Corr}(\mathrm{rule}_n)$ at w_n.

23 Another way of making the point is that we have, until now, been assuming something like the claim that whether a particular translation rule is correct is decidable.

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than the result of applying rule [6] (or, given the arguments made above, perhaps rule [5]).

This is a real issue, and one that deserves more attention. That being said, however, we will set it aside here, and assume for the remainder of this essay that there is a correct translation rule (and that, whatever it turns out to be, it is weaker than rule [1]). Assuming that there is a single correct rule for translating informal intuitionistic assertions containing "unless" into our formal language will simplify the discussion in the remainder of this essay. In addition, I see no reason for thinking that any of the points made below regarding logical revision depend on this assumption, but making this assumption will greatly simplify the making of these points.[24]

4 "Unless" and Logical Revision

So, what is the upshot of all of this? Why does it *matter* how an intuitionist translates "unless," and how such translations might differ from the way classical logicians translate the same bit of natural language? To begin to develop the answer to this question, we again need to think about how logic is taught in introductory formal logic courses, this time with an eye towards the *order* in which various skills are introduced.

In most introductory logic courses, and in most texts on which such courses are based, the topics in question are introduced in roughly the following order:[25]

1. Students are introduced to a particular formal language (e.g. the language of propositional logic).
2. Students are taught how to translate informal natural language sentences and arguments into the formal language, and vice versa.

24 In addition, the close connections drawn by intuitionists between the meaning of expressions and our *manifested* use of those expressions—see Dummett (1975) for a classic source—makes the assumption that there is a unique correct translation rule rather plausible.

25 Of course, in most real-world introductory courses that fit the pattern I have described, the third step involves introducing students to a single account of logic and implicitly assuming (for the sake of the course, at least) that this logic—classical logic—is correct (and hence that the translation rules given in the second step are also correct). But notice that the pattern is the same in textbooks on non-classical logics. See, for example, Sider (2010), where formalization is introduced in chapter 1, long before either classical or non-classical deductive systems or semantics are introduced (in chapters 2 and 3 respectively).

3. Students are taught how to evaluate the sentences and arguments in the formal language (e.g. in terms of logical truth/falsity, validity/invalidity) via either a deductive system or a formal semantics or both.

In short, on the way that formal logic is usually taught, the correct rules for translating natural language sentences and arguments into our formal language is prior to, and hence *must be independent of*, the introduction of the logic via which we shall evaluate those arguments.

Now, from a pedagogical perspective, this might well be the best way to introduce these topics. But once we are engaged in arguments regarding the correct logic, this gets things exactly backwards. As we have seen, the correct translation of "unless" into formal languages depends on which logic one is using—translating "unless" as "or" is perfectly acceptable if one is a classical logician, but is deeply mistaken if one is an intuitionistic logician. And—and this is the rub—this observation has ramifications for how we carry out debates regarding logical revision.

We can flesh out the point by considering a somewhat contrived variant on a classic argument for logical revision due to Hilary Putnam, based on the famous double-slit experiment.[26] In this experiment, photons are projected so that they pass through a plate with two slits cut into it and then collide with a detection screen. When the photons are projected through the plate without any observation regarding the slit through which they passed, the resulting pattern of impacts on the detection screen displays an *interference pattern* associated with wavelike behavior, and seemingly incompatible with each photon having traveled particle-like through exactly one or the other of the slits.

Given this (admittedly rather informal) description of the double-slit experiment, assume that we fire some photons, one-at-a-time, through the apparatus and we observe the expected interference pattern. Then, letting p be any one of the photons, consider the following claims:

1. p impacted the detection screen at location λ, and p passed through the first slit, unless it passed through the second slit.

26 I am merely using this example, and the physics underlying the example, to illustrate the general methodological issue I wish to raise with regard to debates about logical revision. Thus, I will describe the details briefly and somewhat simplistically. Readers interested in more a more careful discussion of Putnam's argument and assessments of its success should consult the extensive literature on this topic, which includes Gardner (1971), Dummett (1976), Gibbins (1987), Hellman (1981), and Maudlin (2005).

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2. Either p impacted the detection screen at location λ and passed through the first slit, or p impacted the detection screen at location λ and passed through the second slit.

Putnam (in effect—he of course does not use "unless" in constructing his version of the argument) argues that physics tells us that the first claim is true, and the second claim fails to be true.[27] Let's grant this much. Now, adopting the following translation manual:

$A =_{df}$ p impacted the detection screen at location λ

$B_1 =_{df}$ p passed through the first slit

$B_2 =_{df}$ p passed through the second slit

the classical logician will formalize these claims as (or as something equivalent to):

1. $A \wedge (B_1 \vee B_2)$
2. $(A \wedge B_1) \vee (A \wedge B_2)$

Putnam also points out that the latter follows from the former in any logic L that accepts the following instance of the distributivity rule:

$$\Phi \wedge (\Psi_1 \vee \Psi_2) \vdash_{\mathrm{L}} (\Phi \vee \Psi_1) \vee (\Phi \vee \Psi_2)$$

Now, classical logic accepts the distributivity rule. Thus, if Putnam is right about the physics, then we must abandon classical logic, and replace it with a logic (such as the quantum logic Q that Putnam is endorsing) that (at a minimum) fails to validate this instance of distributivity. Since we agreed, for the sake of the example, to accept that Putnam is right about the physics, so much for classical logic. We must revise.

But what about the intuitionist? After all, the relevant distributivity law is also valid in intuitionistic logic. Does it follow that the intuitionist, like the classical logician, needs to revise their logic, abandoning intuitionistic logic for Q (or perhaps some constructive variant of it)?

27 Note the careful wording. Given that we are comparing classical logic and intuitionistic logic, we need to take care to distinguish between claims that are false and those that (in the relevant intuitionistic sense) merely fail to be true.

By this point it will surely come as no surprise to the reader to discover that the answer is "of course not." The intuitionist has another move available to her at this point. Instead of rejecting distributivity, and intuitionistic logic with it, the intuitionist can instead reject the translation manual used by the classical logician in rendering the informal claims about the physics into formal language. With the points of the previous two sections in mind, she can instead insist that we abandon the faulty rule [1], and instead adopt one of rules [2] through [6] as the proper way to translate "unless" claims. And, although we argued above that [4] (or, perhaps, [5], if one *really* wants commutativity) is the correct rule for translating intuitionistic "unless" claims, it turns out that, in this example, any of rules [2] through [6] will do. Given any of these translation manuals, the translation of the antecedent *does not* entail the translation of the consequent. Since rule [2] provides the strongest translation, it is enough to note that:

$$A \wedge ((\neg B_1 \rightarrow B_2) \wedge (\neg B_2 \rightarrow B_1)) \nvdash_{\mathrm{H}} (A \wedge B_1) \vee (A \wedge B_2)$$

Of course, this is an extremely contrived example.[28] But the lesson we can learn from it is not—it is completely general, and of deep significance, for debates about the correct logic.

Given the way that logic is taught, it is perhaps natural to think that translating informal natural language into formal languages is logic-neutral. As a result, it is tempting to think that the right way to evaluate a purported counterexample to some class of logics (i.e. an argument where the premises are true, the conclusion fails to be true, and the argument is valid according to the logics under consideration) is to first give such a univocal, logic-neutral translation into symbols, and then evaluate the validity of the resulting formal argument pattern with respect to whatever logics are under consideration, rejecting those logics that validate the argument, and accepting one (or perhaps more, if one is a pluralist of some sort) of those that do not. In short, it is natural to accept the following schema—which we shall call the *Flawed Argument for Revising Logic* (or FARL)—as correctly describing much of what goes on in debates about logical revision:

THE (FLAWED) ARGUMENT FOR REVISING LOGIC.

28 It is based upon a far less contrived example. See Cook (2018) for a general discussion of Putnam's example and translation into intuitionistic logic—a discussion that does not depend upon anything particular to "unless." The current essay can be seen as a companion piece to that essay.

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(Prem$_1$) We have evidence in favor of accepting natural language claim $\Phi_{\mathcal{NL}}$.
(Prem$_2$) We have evidence in favor of rejecting natural language claim $\Psi_{\mathcal{NL}}$.
(Prem$_3$) Within the context of our current formal logic L_1, $\Phi_{\mathcal{NL}}$ is best translated as Φ_{L_1}.
(Prem$_4$) Within the context of our current formal logic L_1, $\Psi_{\mathcal{NL}}$ is best translated as Ψ_{L_1}.
(Prem$_5$) The argument from Φ_{L_1} to Ψ_{L_1} is valid in our current formal logic L_1, that is:

$$\Phi_{\mathsf{L}_1} \vdash_{\mathsf{L}_1} \Psi_{\mathsf{L}_1}$$

(Conc) We should abandon formal logic L_1 in favor of a weaker (or at least different) logic L_2 where:

$$\Phi_{\mathsf{L}_1} \nvdash_{\mathsf{L}_2} \Psi_{\mathsf{L}_1}$$

But the conclusion does not follow from the premises. After all, why should we think, as is required by the conclusion Conc, that we need to move to a new logic L_2 that does not validate the inference whose premise is the correct translation of $\Phi_{\mathcal{NL}}$, and whose conclusion is the best translation of $\Psi_{\mathcal{NL}}$, *where correctness is understood as relative to our old, now rejected, logic* L_1? Of course, if translation from natural language to formal language were logic-neutral, so that the correct translation of these claims from the perspective of L_1 just was the best translation of these claims from the perspective of L_2, then this wouldn't matter. But, as we now know, translation is not logic neutral. Thus, the conclusion of the argument pattern given above should instead be:

(Conc) We should abandon formal logic L_1 in favor of a weaker (or at least different) logic L_2 where:

$$\Phi_{\mathsf{L}_2} \nvdash_{\mathsf{L}_2} \Psi_{\mathsf{L}_2}$$

(and where Φ_{L_2} and Ψ_{L_2} are the best translations of $\Phi_{\mathcal{NL}}$ and $\Psi_{\mathcal{NL}}$, respectively, from the perspective of L_2.)

Let us call this improved argument pattern, consisting of the premises of FARL and this new conclusion, the *Corrected Argument for Revising Logic* (or CARL).

Thus, if we currently accept a particular logic L, and are then presented with a natural language argument where we accept the premises, we reject the

conclusion, and the translation of the premises into our formal logic (where the correctness of the translation is judged from the perspective of our current logic L) entail the translation of the conclusion into our formal language (again, where translation is judged from the perspective of L), then we have not one but two possible strategies:

1. Switch to a logic where the offending inference is no longer valid.
2. Switch to a logic where the correct translations of the premises and conclusion are different.

In our toy example, the logician who rejects classical logic C in favor of quantum logic Q is adopting the first option (assuming that the correct translation of the premise and conclusion is the same from the perspective of C and from the perspective of Q). The classical logician who instead shifts to intuitionistic logic H (or the intuitionist logician who makes no changes to her logic) and rejects the disjunctive translation of the premises is instead adopting the second strategy.

Of course, this is, as I have emphasized repeatedly, a somewhat contrived example.[29] Nevertheless, the lesson it teaches us is deep, and can be summarized as follows:

- A particular counterexample *C* (of the sort described in the premises of FARL or CARL) can show us that a particular logic L must be rejected.
- A particular counterexample *C* (of the sort described in the premises of FARL or CARL) can never, on its own, show us that a particular inferential pattern or rule is invalid.

For any particular inference rule which seems to be challenged by a counterexample in the way that Putnam's quantum logic example seems to challenge the distributivity laws, we are (at least, in principle) free to adopt a logic that retains that rule, as long as, from the perspective of that logic, the correct translation of the premise(s) and conclusion of the purported counterexample no longer instantiate the rule in question. Of course, moving to such a logic, instead of moving to a logic where the inference rule is no longer valid, will not always be the right move, or even a plausible one (for example, it

29 To emphasize: I am not suggesting that the *right* move, for the logician faced with Putnam's purported counterexample, is the one suggested here. Instead, the point is merely that it *is* a move, and, further, there will no doubt be genuine (non-contrived) cases where it is the right move.

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would be absurd for someone sympathetic to Dummett-style worries about excluded middle to retain classical logic, but argue that all natural language expressions of the form $\Phi \vee \neg\Phi$ should be translated as a random contingent sentence—e.g. Φ itself). But there will be some cases where this is the right move, and realizing this requires that one recognize that translation from natural language to formal languages (and vice-versa) is not logic-neutral.

5 Conclusion

We'll conclude the paper by explaining its title. First, we can flesh out its content a bit more:

> Unless "$\neg A$ unless A" is invalid, "A unless B" is equivalent to "A or B."

We can now make this more formal along the following lines. For the classical logician applying translation rule [1], this becomes:

$$\text{Either: } \nvDash_{\mathrm{C}} \neg A \vee A \text{ or: } A \vee B \dashv\vdash_{\mathrm{C}} B \vee A$$

The right-hand-side of this disjunction (hence the disjunction as a whole) is obviously classically true. If the arguments given here are correct, however, the intuitionist should apply translation rule [4], and understand this claim as:[30]

$$\text{If not: } \nvDash_{\mathrm{H}} \neg A \rightarrow \neg A \text{ then: } \neg B \rightarrow A \dashv\vdash_{\mathrm{H}} \neg A \rightarrow B$$

Now, the antecedent of this conditional is true, via an intuitionistically valid application of double negation introduction in the metalanguage to obtain:

$$\text{not: } \nvDash_{\mathrm{H}} \neg A \rightarrow \neg A$$

The consequent of this conditional is clearly false, however. Thus, the conditional as a whole is intuitionistically false.[31]

This brings up a final issue that, again, for the sake of short(ish)ness and snappy(ish)ness, we will only be able to touch on briefly here. There is a

30 Examination of the title of the paper from the perspective of rules [2], [3], [5], and [6] is left to the interested reader.

31 As a result, this is probably the first time I have given a paper a title that I believe (due to my own intuitionistic leanings) is false!

substantial debate within the philosophy of logic concerning what has come to be called the "communication problem"—that is, on determining whether intuitionistic and classical logicians mean the same thing by "and," "or," "not," etc., and are just disagreeing about which claims involving these expressions are valid; or whether they mean different things by these expressions and hence are failing, in some sense, to be disagreeing (or even communicating at all) with each other.[32] I have long been sympathetic to the former understanding, and I am not alone.[33] But the arguments presented above seem to throw some doubt on that understanding of the debate. The difference between the classical and the intuitionistic understanding of the title of this paper does not seem to be merely a difference in the truth value they assign to the claim in question—on the contrary, it seems (at least, intuitively) as if they *mean* different things.

This, in turn, is explained by the fact that the intuitionist and the classical logician cannot both mean the same thing by "or" and mean the same thing by "unless." Assume for *reductio* that they did. Then, since meaning determines truth conditions, then they would assign the same truth conditions to "or" and to "unless." But, by the transitivity of sameness of truth conditions (and the fact that the classical logician assigns the same truth conditions to "unless" and to "or"), it should follow that the intuitionist assigns the same truth conditions to "unless" and to "or." But as we have seen, they do not. Thus, it can't be the case that intuitionists and classical logicians have a shared set of meanings for all of the logical expressions in natural language. Unfortunately, an in-depth examination of this issue will have to wait for another time.*

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32 For a good discussion of this debate, see Hellman (1989).

33 See e.g. Tennant (1996) for an account of intuitionism that seems to depend on shared meanings.

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References

Brouwer, Luitzen Egbertus Jan. 1954. "Points and Spaces." *Canadian Journal of Mathematics* 6: 1–17, doi:10.4153/CJM-1954-001-9.

Cook, Roy T. 2018. "Logic, Counterexamples, and Translation." In *Hilary Putnam on Logic and Mathematics*, edited by Geoffrey Hellman and Roy T. Cook, pp. 17–44. Outstanding Contributions to Logic n. 9. Cham: Springer Nature.

Dummett, Michael A. E. 1975. "The Philosophical Basis of Intuitionistic Logic." In *Logic Colloquium '73*, edited by H. E. Rose and John C. Shepherdson, pp. 5–40. Studies in Logic and the Foundations of Mathematics n. 80. Amsterdam: North-Holland Publishing Co. Reprinted in Dummett (1978, 215–247).

—. 1976. "Is Logic Empirical?" In *Contemporary British Philosophy, 4th series*, edited by Hywel David Lewis, pp. 45–68. London: George Allen & Unwin. Reprinted in Dummett (1978, 269–289).

—. 1977. *Elements of Intuitionism*. Oxford Logic Guides n. 2. Oxford: Oxford University Press.

—. 1978. *Truth and Other Enigmas*. Cambridge, Massachusetts: Harvard University Press.

Gardner, Michael R. 1971. "Is Quantum Logic Really Logic?" *Philosophy of Science* 38(4): 508–529, doi:10.1086/288393.

Gibbins, Peter F. 1987. *Particles and Paradoxes. The Limits of Quantum Logic*. Cambridge: Cambridge University Press.

Hellman, Geoffrey. 1981. "Quantum Logic and Meaning." In *PSA 1980: Proceedings of the Biennial Meeting of the Philosophy of Science Association, Part II: Symposium Papers*, edited by Peter D. Asquith and Ronald N. Giere, pp. 493–511. East Lansing, Michigan: Philosophy of Science Association.

—. 1989. "Never say 'Never'! On the Communication Problem between Intuitionism and Classicism." *Philosophical Topics* 17(2): 47–67, doi:10.5840/philtopics19891723.

Higginbotham, James. 1986. "Linguistic Theory and Davidson's Program in Semantics." In *Truth and Interpretation: Perspectives on the Philosophy of Donald Davidson*, edited by Ernest LePore, pp. 29–48. Oxford: Basil Blackwell Publishers.

Humberstone, I. Lloyd. 2011. *The Connectives*. Cambridge, Massachusetts: The MIT Press.

Kreisel, Georg. 1962. "On Weak Completeness of Intuitionistic Predicate Logic." *The Journal of Symbolic Logic* 27(2): 139–158, doi:10.2307/2964110.

Maudlin, Tim. 2005. "The Tale of Quantum Logic." In *Hilary Putnam*, edited by Yemima Ben Menahem, pp. 156–187. Contemporary Philosophy in Focus. Cambridge: Cambridge University Press, doi:10.1017/UPO9781844653119.

Putnam, Hilary. 1969. "Is Logic Empirical?" In *Proceedings of the Boston Colloquium for the Philosophy of Science 1966/1968*, edited by Robert S. Cohen and Marx W.

WARTOFSKY, pp. 216–241. Boston Studies in the Philosophy of Science n. 5. Dordrecht: D. Reidel Publishing Co. Reprinted as "The Logic of Quantum Mechanics" in Putnam (1975, 174–197).

—. 1975. *Mathematics, Matter and Method. Philosophical Papers, Volume 1*. Cambridge: Cambridge University Press. Second edition: Putnam (1979).

—. 1979. *Mathematics, Matter and Method. Philosophical Papers, Volume 1*. 2nd ed. Cambridge: Cambridge University Press. First edition: Putnam (1975).

SIDER, Theodore. 2010. *Logic for Philosophy*. New York: Oxford University Press.

TENNANT, Neil W. 1996. "The Law of Excluded Middle Is Synthetic A Priori, If Valid." *Philosophical Topics* 24(1): 205–230, doi:10.5840/philtopics19962416.

—. 2017. *Core Logic*. Oxford: Oxford University Press, doi:10.1093/oso/9780198777892.001.0001.

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